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DECIDABILITY AND STABILITY OF FREE NILPOTENT LIE ALGEBRAS AND FREE NILPOTENT p-GROUPS OF FINITE EXPONENT

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Our main result is the decidability and ω-stability of free cth nilpotent p-groups of finite exponent ($c < p$).

1. Introduction

Let c be a natural number greater than 1, p a prime greater than c, and κ any cardinal greater than 1. Let $L_c(R, \kappa)$ be the free cth nilpotent Lie algebra over the field R with κ generators. Using the Theorem of Širšov–Witt ([21, 24]), that every subalgebra of a free Lie algebra over a field is free again, we characterize the subalgebras of $L_c(R, \kappa)$ up to isomorphisms by a c-tuple of cardinals.

The group with κ generators free in the variety of all nilpotent groups of class c with exponent p^n will be denoted by $F_c(p^n, \kappa)$. We get a characterization of the subgroups of $F_c(p, \kappa)$, considering the corresponding free nilpotent Lie algebra. Our algebraical results above will be used to give a recursive axiom system $\Sigma_c(p)$ for the elementary theory of $F_c(p, \omega)$. Let $\cdots \geqslant Z_c \geqslant Z_{c-1} \geqslant \cdots \geqslant Z_1 \geqslant \langle 1 \rangle$ denote the upper central series of a group. $\Sigma_c(p)$ consists of:

(1) nilpotency of class c, exponent p, and $[Z_{c+1-i}, Z_{c+1-j}] \subseteq Z_{c+1-(i+j)}$ (for $i+j > c+1$ define $Z_{c+1-(i+j)} = Z_0$);

(2) Z_c/Z_{c-1} is infinite;

(3) elementary description of the structure of finite subgroups.

The models of $\Sigma_c(p)$ are exactly the subgroups G of $F_c(p, \kappa)$ such that $G/Z_{c-1}(G)$ is infinite. $\Sigma_c(p)$ is complete and ω-stable. Therefore $F_c(p, \omega) \equiv F_c(p, \kappa)$ for $\omega \leqslant \kappa$, and $\Sigma_c(p)$ is decidable. For $c = 2$ this is proved by Ershov in [4].

Introducing definable new predicates we show the elimination of quantifiers. Then we extend our results to free cth nilpotent groups of exponent p^n. In contrast to our results Malzew [16] had shown that the elementary theory of a free nilpotent group is hereditarily undecidable. Furthermore from his proof unstability follows. For absolutely free groups these questions are open. It is only known that they are not superstable (Gibone, see [25]). By Ershov [4] and

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Samjatin [20] the elementary theory of every non-abelian variety of groups is undecidable. However, by the results above, there are relatively free groups of some non-abelian variety having a decidable theory. We have the same situation for stability here. Indeed any non-abelian variety contains an unstable member (this was independently proved by Baldwin and Saxl [1] and Belegradek [3]). However, in our varieties the free objects are even ω-stable.

In the last section we consider the elementary theory $\mathrm{Th}(L_c(R,\kappa))$ of the free cth nilpotent Lie algebra over the field R with κ generators in a two-sorted language ($\omega \leqslant \kappa$). The methods developed for groups are applicable to prove the following transfer theorems:

$\mathrm{Th}(L_c(R,\kappa))$ is recursive relative to $\mathrm{Th}(R)$.

$\mathrm{Th}(L_c(R,\kappa))$ and $\mathrm{Th}(R)$ have the same stability class if R is infinite. $\mathrm{Th}(L_c(R,\kappa))$ is ω-stable otherwise. The main step in the proofs of these theorems is the elimination of quantifiers.

I was stimulated to investigate the elementary theory of free nilpotent groups of finite exponent by a paper of Mekler [17]. I gratefully acknowledge helpful conversations with Philipp Rothmaler.

2. Consequences of the Theorem of Širšov–Witt

Let R be a fixed field, and $L(R,\kappa)=L$ be the free Lie algebra over R with κ generators. Suppose $\kappa>1$. We use (x, y) to denote the Lie product operation. As in group theory the lower and upper central series for a Lie algebra M over R are defined:

$\Gamma_1(M)=M$. $\Gamma_n(M)$ is the subalgebra of M generated by all Lie products of the form (x, y) where $x\in\Gamma_{n-1}(M)$ and $y\in M$. $Z_0(M)=\langle 0\rangle$. $Z_n=\{x\in M: \forall y\ (x,y)\in Z_{n-1}(M)\}$. For convenience define $Z_k(M)=Z_0(M)$ for $k<0$.

$\Gamma_n(M)$ and $Z_n(M)$ are ideals in M. M is nilpotent of class $\leqslant c$ iff $\Gamma_{c+1}(M)=0$ iff $Z_c(M)=M$. $Z_n(M)$ is elementarily definable. $L(R,\kappa)/\Gamma_{c+1}(L(R,\kappa))$ is the free cth nilpotent Lie algebra $L_c(R,\kappa)$ over R with κ generators (short L_c).

Let A be a subset of a Lie algebra M over R. We use $\langle A\rangle$ to denote the subalgebra generated by A. As usual we introduce the notion of a *monomial b of A-weight n on A* (notation $w_A(b)=n$): The elements of A are the monomials of A-weight 1 on A. b is a monomial of A-weight n on A iff $b=(b_1,b_2)$, where b_1, b_2 are monomials of A-weight r resp. of A-weight s on A and $r+s=n$.

By induction on the A-weight n we define *basic monomials* on A. Every definition of basic monomials contains an order $<$ of them, such that $w_A(b_1)<w_A(b_2)$ implies $b_1<b_2$. The elements of A are the basic monomials of A-weight 1. Assume that the basic monomials on A of A-weight less than n have already been defined and are ordered according to their definition. b is a basic monomial of A-weight n iff $b=(b_1,b_2)$, where b_1 and b_2 are basic monomials, $w_A(b_1)+w_A(b_2)=n$, $b_1>b_2$, and if $b_1=(b_3,b_4)$, then $b_2\geqslant b_4$.

There are several procedures to obtain basic monomials on A depending on the order defined for each A-weight. In most cases we assume one of them fixed. Independent from this the following hold:

Theorem 2.1 (M. Hall [10]). *If A is a set of free generators of L (L_c), then every element of L (L_c) can be expressed uniquely as a linear combination of basic monomials on A (of A-weight less than $c+1$).*

Studying M. Hall's proof of this theorem we get:

Lemma 2.2. *Assume A is a subset of a Lie algebra M over R. Then every element a of $\langle A \rangle$ can be expressed as a linear combination of basic monomials on A. If b_1 and b_2 are basic monomials on A, then (b_1, b_2) can be expressed as a linear combination $f(c_1, \ldots, c_n)$ of basic monomials c_i on A with coefficients from the prime subfield of R. We can get $f(c_1, \ldots, c_n)$ in a uniform way independent from M. Furthermore we can assume: $d \in A$ occurs m-times in the monomial c_i iff d occurs m-times in (b_1, b_2).*

Širšov [21] and Witt [24] have proved that every subalgebra of a free Lie algebra over a field is free. We give a description of their result useful for our purposes. A similar presentation you find in [18].

Unfortunately Γ_i is not elementarily definable in $L_c(R, \kappa)$. But $\Gamma_i(L_c(R, \kappa)) = Z_{c+1-i}(L_c(R, \kappa))$ for $1 \leq i \leq c+1$, as easily proved using Theorem 5.10 in [15, p. 328]. This enables us to use the members Z_i of the upper central series instead. Let M be a cth nilpotent Lie algebra over R with the following property:

(Z) $\qquad (Z_{c+1-i}(M), Z_{c+1-j}(M)) \subseteq Z_{c+1-(i+j)}(M) \quad \text{for } 1 \leq i, j \leq c+1.$

Let $A = \{a^i_{\alpha(i)} : 1 \leq i \leq c, \alpha(i) < \lambda_i\}$ be a subset of M. In order to define basic monomials on A we use the following order of A: $a^i_\alpha < a^j_\beta$ iff $i < j$ or $i = j$ and $\alpha < \beta$. In addition to A-weight we introduce the *A-degree* $d_A(b)$ of monomials b on A: $d_A(a^i_{\alpha(i)}) = i$, $d_A((b_1, b_2)) = d_A(b_1) + d_A(b_2)$. A is called an *(o)-system* iff $a^i_{\alpha(i)} \in Z_{c+1-i}(M)$ and for every $1 \leq n \leq c$ $\{a^n_{\alpha(n)} : \alpha(n) < \lambda_n\}$ is linearly independent modulo the ideal generated by $Z_{c-n}(M)$ and all basic monomials on $\{a^j_{\alpha(j)} \in A : j < n\}$ of A-degree n. A is called a *$(*)$-system* iff $a^i_{\alpha(i)} \in Z_{c+1-i}(M)$ and for every $1 \leq n \leq c$ the set of all basic monomials on A of A-degree n is linearly independent modulo $Z_{c-n}(M)$.

In the next lemma we list some properties of (o)- and $(*)$-systems, easily to prove:

Lemma 2.3. *Let M be a cth nilpotent Lie algebra over a field with the property (Z). Then*

(i) *for every subalgebra of M there is a generating (o)-system.*

(ii) *every $(*)$-system is an (o)-system.*

(iii) *A is an (o)-system $((*)$-system$)$ in M iff every finite subset of A is an (o)-system $((*)$-system$)$ in M.*

Now we are able to formulate the Theorem of Širšov–Witt useful for our model-theoretic purposes.

Theorem 2.4 (Širšov [21], Witt [24]). *In $L_c(R, \kappa)$ every (o)-system is a $(*)$-system.*

Proof. We derive our special version of the Theorem of Širsov–Witt from the Key-lemma 5 in [21, p. 448]. According to $\Gamma_i(L_c(R,\kappa)) = Z_{c+1-i}(L_c(R,\kappa))$ we use the upper central series. If X is a set of free generators of $L = L(R,\kappa)$ we have $L = \bigoplus_{1 \leq n < \omega} L^n$ where L^n is the submodule generated by all monomials on X of X-weight n [15], p. 301, Lemma 5.1. Therefore for each $a \in L$ there are elements $a_{i_1}, \ldots, a_{i_s}$ such that $a_{i_j} \in L^{i_j}$ and $a = \sum_{1 \leq j \leq s} a_{i_j}$. The a_{i_j} are uniquely determined. a is homogenous iff $a = a_{i_1} \in L^{i_1}$. Širšov proved in [21, Lemma 5, p. 448] the following:

(S) Let $C = \{c_1, \ldots, c_n\}$ be a finite subset of homogenous elements of L such that for every i c_i is not an element of the subalgebra $\langle\{c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n\}\rangle$.
Then we have $f(c_1, \ldots, c_n) \neq 0$ for every nontrivial linear combination $f(c_1, \ldots, c_n)$ of basic monomials on C.

Let $A = \{a^i_{\alpha(i)}: 1 \leq i \leq c, \alpha(i) < \lambda_i\}$ be a subset of L such that the image $\bar{A}$ of A in $L_c = L/\Gamma_{c+1}(L)$ is not a $(*)$-system of L_c. Then for some n $(1 \leq n \leq c)$ $f(a_1, \ldots, a_s) \in \Gamma_{n+1}(L)$, where $f(a_1, \ldots, a_s)$ is a linear combination of basic monomials on a subset $\{a_1, \ldots, a_s\}$ of A of A-degree n.

W.l.o.g. we can assume that $a_1, \ldots, a_s$ are homogenous. Then $f(a_1, \ldots, a_s) = 0$. (S) implies $a_i = g(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_s)$ for some i, where $g(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_s)$ is a linear combination of basic monomials on $\{a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_s\}$ of A-degree $d_A(a_i)$. Therefore $\bar{A}$ is not an (o)-system, as desired. $\square$

Lemma 2.5. *Let $A = \{a^i_{\alpha(i)}: 1 \leq i \leq c, \alpha(i) < \lambda_i\}$ be a $(*)$-system of a cth nilpotent Lie algebra M over the field R with the property (Z). Let $\{b_\alpha : \alpha < \mu\}$ be an enumeration of all basic monomials on A ordered according to A-degree. Then every element g of $\langle A\rangle$ can be expressed uniquely as $g = \Sigma_\alpha r_\alpha b_\alpha$ where $r_\alpha \in R$ and $r_\alpha = 0$ up to finitely many α. If $g \in Z_{c+1-n}(M) \cap \langle A\rangle$, then $r_\alpha = 0$ for all α such that $d_A(b_\alpha) < n$.*

Proof. Let $\{b_\alpha : \alpha < \mu_n\}$ be the set of all basic monomials on A of A-degree $\leq n$. First of all we prove:

$$\{b_\alpha : \mu_n \leq \alpha < \mu\} \text{ generates } Z_{c-n}(M) \cap \langle A\rangle.$$

Let g be any element of $Z_{c-n}(M)\cap\langle A\rangle$. By Lemma 2.2 $g=\Sigma_\alpha r_\alpha b_\alpha$, where $r_\alpha\in R$ and $r_\alpha=0$ up to finitely many α. Let $b_{\alpha_1},\dots,b_{\alpha_m}$ be the monomials in this presentation of g with minimal A-degree and $r_{\alpha_i}\neq 0$. There is some l with $\mu_{l-1}\leqslant\alpha_i<\mu_l$. Since A is a (*)-system

$$\Big(\sum_{1\leqslant i\leqslant m} r_{\alpha_i}b_{\alpha_i}\neq 0\Big)\quad\text{modulo } Z_{c-l}(M).$$

Hence $g\neq 0$ modulo $Z_{c-l}(M)$. This implies $n<l$, as desired. Now it is easy to prove the assertion by induction on c. Apply the induction hypothesis to $M/Z_1(M)$. □

Immediate consequences of Theorem 2.4 and Lemma 2.5 are:

Corollary 2.6. *Let $A=\{a^i_{\alpha(i)}: 1\leqslant i\leqslant c, \alpha(i)<\lambda_i\}$ be a subset of $L_c(R,\kappa)$ with $a^i_{\alpha(i)}\in Z_{c+1-i}(L_c(R,\kappa))$.*

(i) *A is an (o)-system in $L_c(R,\kappa)$ iff for every n $(1\leqslant n\leqslant c)$ $\{a^n_{\alpha(n)}\in A:\alpha(n)<\lambda_n\}$ is linearly independent modulo the ideal generated by $Z_{c-n}(L_c(R,\kappa))$ and*

$$Z_{c+1-n}(L_c(R,\kappa))\cap\langle\{a^j_{\alpha(j)}\in A: 1\leqslant j<n\}\rangle.$$

(ii) *If $\mathfrak{S}_i$ is a permutation of $\{\alpha:\alpha<\lambda_i\}$, then A is an (o)-system of $L_c(R,\kappa)$ iff $\{a^i_{\mathfrak{S}_i(\alpha(i))}: 1\leqslant i\leqslant c, \alpha(i)<\lambda_i\}$ is an (o)-system of $L_c(R,\kappa)$.*

Corollary 2.7. *Let $A=\{a^i_{\alpha(i)}: 1\leqslant i\leqslant c, \alpha(i)<\lambda_i\}$ be an (o)-system of $L_c(R,\kappa)$. Let $\{b_\alpha:\alpha<\mu\}$ be an enumeration of all basic commutators on A ordered according to A-degree. Then every element g of $\langle A\rangle$ can be expressed uniquely as $g=\sum_\alpha r_\alpha b_\alpha$, where $r_\alpha\in R$ and $r_\alpha=0$ up to finitely many α.*

Corollary 2.8. *Let $A=\{a_\alpha:\alpha<\lambda\}$ be a set of elements of $Z_{c+1-n}(L_c(R,\kappa))$ linearly independent modulo $Z_{c-n}(L_c(R,\kappa))$. Then A freely generates a subalgebra isomorphic to $L_{[c/n]}(R,\kappa)$, where $[c/n]$ denotes the integral part of c/n.*

Corollary 2.9. *If $A=\{a^i_{\alpha(i)}: 1\leqslant i\leqslant c, \alpha(i)<\lambda_i\}$ and $C=\{c^i_{\alpha(i)}: 1\leqslant i\leqslant c, \alpha(i)<\lambda_i\}$ are two (o)-systems in $L_c(R,\kappa)$, then $\varphi(a^i_{\alpha(i)})=c^i_{\alpha(i)}$ induces an isomorphism of $\langle A\rangle$ onto $\langle C\rangle$.*

Proof. By Lemma 2.2 there is a uniform way to bring some element of $\langle A\rangle$ resp. $\langle C\rangle$ into the unique form of Lemma 2.5. □

Lemma 2.10. *Let H be a subalgebra of $L_c(R,\kappa)$ and $A=\{a^i_{\alpha(i)}: 1\leqslant i\leqslant c, \alpha(i)<\lambda_i\}$ and $C=\{c^i_{\alpha(i)}: 1\leqslant i\leqslant c, \alpha(i)<\mu_i\}$ be two generating (o)-systems of H. Then $\lambda_i=\mu_i$ for $1\leqslant i\leqslant c$.*

Proof. Since $Z_{c+1-i}(L_c)=\Gamma_i(L_c)$ we use $\Gamma_i(L_c)$ and write Γ_i only. It is sufficient to show that for every $n\leq c$ every basic monomial b on $\{a^i_{\alpha(i)}\in A: i<n\}$ of A-degree n is an element of $\langle\{c^i_{\alpha(i)}\in C: i<n\}\rangle$ modulo Γ_{n+1}. Assume b is chosen as above. Then $b=(b_1,b_2)$, where b_1 and b_2 are basic monomials on $\{a^i_{\alpha(i)}\in A: i<n\}$, $b_1\in\Gamma_r$, $b_2\in\Gamma_s$, and $r+s=n$. Since $r,s<n$ b_1 is an element of $\langle\{c^i_{\alpha(i)}\in C: i<n\}\rangle\cap\Gamma_r$ modulo Γ_{r+1}, and b_2 is an element of $\langle\{c^i_{\alpha(i)}\in C: i<n\}\rangle\cap\Gamma_s$ modulo Γ_{s+1} (Lemma 2.5). It follows the assertion. □

By Lemma 2.3(i), Corollary 2.9, and Lemma 2.10 we obtain:

Theorem 2.11. *The subalgebras H of $L_c(p,\kappa)$ are characterized up to isomorphisms by the parameter λ_i $(1\leq i\leq c)$ of any generating (o)-system $A=\{a^i_{\alpha(i)}: 1\leq i\leq c, \alpha(i)<\lambda_i\}$ of H.*

We say that H is a $(\lambda_1,\ldots,\lambda_c)$-*subalgebra* of $L_c(R,\kappa)$. Similarly as Corollary 2.9 we prove:

Corollary 2.12. *Let $A=\{a^i_{\alpha(i)}: 1\leq i\leq c, \alpha(i)<\lambda_i\}$ be a $(*)$-system in a cth nilpotent Lie algebra over R with (Z). Then $\langle A\rangle$ is isomorphic to every $(\lambda_1,\ldots,\lambda_c)$-subalgebra M of every $L_c(R,\kappa)$ where $\kappa\geq\lambda_i$. If $C=\{c^i_{\alpha(i)}: 1\leq i\leq c, \alpha(i)<\lambda_i\}$ is a generating (o)-system of M, then the desired isomorphism φ is induced by $\varphi(a^i_{\alpha(i)})=c^i_{\alpha(i)}$.*

Lemma 2.13. *Let H be a $(\lambda_1,\ldots,\lambda_c)$-subalgebra of $L_c(R,\kappa)$ such that $\lambda_1\geq 2$. Then $Z_i(L_c(R,\kappa))\cap H=Z_i(H)$.*

Proof. By induction on i one can show $Z_i(L_c)\cap H\subseteq Z_i(H)$. The other direction we prove by induction on c. Then it is sufficient to show that $Z_1(L_c)\cap H\supseteq Z_1(H)$. Let $a\in Z_1(H)$. If $a\notin Z_1(L_c)$, then $a\in Z_m(L_c)\setminus Z_{m-1}(L_c)$, where $m>1$. Since $\lambda_1\geq 2$, there is some $b\in(L_c\setminus Z_{c-1}(L_c))\cap H$, such that $\{a,b\}$ is an (o)-system and by Theorem 2.4 a $(*)$-system. Then $(a,b)\in(Z_{m-1}(L_c)\setminus Z_{m-2}(L_c))\cap H$. Therefore $(a,b)\neq 0$, hence $a\notin Z_1(H)$, a contradiction. □

Lemma 2.13 has an important consequence.

Corollary 2.14. *Let H be a $(\lambda_1,\ldots,\lambda_c)$-subalgebra of some $L_c(R,\kappa)$ with $\lambda_1\geq 2$. Then a subset A of H is an (o)-system ($(*)$-system) in H iff it is an (o)-system ($(*)$-system) in L_c.*

Let $A=\{a^i_{\alpha(i)}: 1\leq i\leq c, \alpha(i)<\lambda_i\}$ be an (o)-system in a subalgebra M of some $L_c(R,\kappa)$. A is called an $(o)_{j,n}$-*system* iff $\lambda_l\leq n$ and $\lambda_i=0$ for $i\geq j$. We write $o_{j,n}(a)$ iff $a\in Z_{c+1-j}(M)$ and there is an $(o)_{j,n}$-system A such that a is an element of $\langle A\rangle$ modulo $Z_{c-j}(M)$. Define $o_{j,0}(X)=Z_{c-j}(x)$.

Lemma 2.15. *Let M be a $(\lambda_1, \dots, \lambda_c)$-subalgebra of some $L_c(R, \kappa)$ with $\lambda_1 \geqslant \omega$, $\kappa \geqslant \omega$. For every $1 < j \leqslant c$, every $n < \omega$, and any given elements $c_1, \dots, c_m$ of $Z_{c+1-j}(M)$ there is some $a \in Z_{c+1-j}(M)$ such that $\neg o_{j,n}(a - \sum_{1 \leqslant i \leqslant m} r_i c_i)$ for all $r_i \in R$.*

Proof. Let $A = \{a^i_{\alpha(i)} : 1 \leqslant i \leqslant c, \alpha(i) < \lambda_i\}$ be a generating (o)-system of M in $L_c(R, \kappa)$. Let $X = \{x_\alpha : \alpha < \kappa\}$ be a set of free generators of $L_c(R, \kappa)$ with $\{a^1_{\alpha(1)} \in A : \alpha(1) < \lambda_1\} \subseteq X$. W.l.o.g. we can suppose the existence of some l with $c_1, \dots, c_m \in \langle\{x_\alpha : \alpha < l\}\rangle$.

(1) Then every $a \in \langle\{a^1_{\alpha(1)} \in A : \alpha(1) < \lambda_1\}\rangle \cap Z_{c+1-j}(M)$ with $\neg o_{j,jn+l}(a)$ in L_c fulfils the assertion.

Otherwise $(a - \sum_{1 \leqslant i \leqslant m} r_i c_i) \in \langle C\rangle$ modulo $Z_{c-j}(M)$ for some $(o)_{j,n}$-system C in M and $r_i \in R$. By Lemma 2.13 C is an $(o)_{j,n}$-system in L_c and we get the same situation in L_c. Then $a \in \langle\{x_\alpha : \alpha < l\} \cup C\rangle$ modulo $Z_{c-j}(L_c)$. This gives a contradiction because we can generate $\langle\{x_\alpha : \alpha < l\} \cup C\rangle$ by an $(o)_{j,jn+l}$-system modulo $Z_{c-j}(L_c)$ in L_c.

By (1) we can assume w.l.o.g. that $m = 0$ and $M = L_c$. We prove the lemma by induction on j. Firstly we consider the case $j = 2$. Let $A = \{a_i : i < 2k\}$ be a set of elements of L_c linearly independent modulo $Z_{c-1}(L_c)$, where $k > \frac{1}{2}n(n-1)$. Then the assertion follows from

(2) $\neg o_{2,n}(a)$ for $a = \sum_{i<k} (a_{2i}, a_{2i+1})$.

Otherwise there would be some $(o)_{2,n}$-system $B = \{b_0, \dots, b_{n-1}\}$ such that $a \in \langle B\rangle$ modulo $Z_{c-z}(L_c)$. There are subsets $B_1 \subseteq B$ and $A_1 \subseteq A$ such that the images $\bar{C}$ and $\bar{D}$ of $C = A \cup B_1$ and $D = B \cup A_1$ in $L_c/Z_{c-1}(L_c)$ are bases of the image $\overline{\langle A \cup B\rangle}$ of $A \cup B$ in $L_c/Z_{c-1}(L_c)$. Let $\bar{C} = \{\bar{c}_0, \dots, \bar{c}_{h-1}\}$ and $\bar{D} = \{\bar{d}_0, \dots, \bar{d}_{h-1}\}$ where $c_0 = a_1, \dots, c_{2k-1} = a_{2k-1}$, and $d_0 = b_1, \dots, d_{n-1} = b_{n-1}$. Then $\varphi(\bar{c}_i) = \bar{d}_i$ induces an automorphism of $\overline{\langle A \cup B\rangle}$. It follows

$$a = \sum_{i<k} (\varphi^{-1}(d_{2i}), \varphi^{-1}(d_{2i+1})) \quad \text{modulo } Z_{c-2}(L_c).$$

We can express φ^{-1} by a matrix $(s_{i,l})$ where $s_{i,l} \in R$ and $\varphi^{-1}(\bar{d}_i) = \sum_{l<h} s_{i,l}\bar{d}_l$. Since φ^{-1} is an automorphism there is a one-to-one function f from $\{i : i < h\}$ onto $\{i : i < h\}$ with

$$s_{2i,f(2i)}s_{2i+1,f(2i+1)} - s_{2i,f(2i+1)}s_{2i+1,f(2i)} \neq 0.$$

Then

$$\bar{a} = \sum_{i<k} \left(\sum_{l<h} s_{2i,l}\bar{d}_l, \sum_{l<h} s_{2i+1,l}\bar{d}_l\right) \quad \text{modulo } Z_{c-2}(L_c).$$

Therefore

$$\bar{a} = \left(\sum_{i<k} r_i(\bar{d}_{f(2i)}, \bar{d}_{f(2i+1)})\right) + \bar{b} \quad \text{modulo } Z_{c-2}(L_c),$$

where

$$r_i = s_{2i,f(2i)}s_{2i+1,f(2i+1)} - s_{2i,f(2i+1)}s_{2i+1,f(2i)} \neq 0,$$

and $\bar{b}$ is a linear combination of elements $(\bar{d}_u, \bar{d}_v)$ with $\{u, v\} \neq \{f(2i), f(2i+1)\}$. This gives a contradiction, since $\bar{a} \in \langle \bar{B} \rangle$ implies

$$\bar{a} = \sum_{i<l<n} t_{i,l}(\bar{d}_i, \bar{d}_l) \quad \text{modulo } Z_{c-2}(L_c)$$

for some $t_{i,l} \in R$ (Lemma 2.5).

To show the assertion for $j > 2$ let $A = \{a_0^1, \ldots, a_{k-1}^1, a_0^{j-1}, \ldots, a_{k-1}^{j-1}\}$ be an (o)-system, where the a_i^{j-1}'s are chosen by the induction hypothesis such that

$$\neg o_{j-1,jn+k}\Big(\sum_{i<k} r_i a_i^{j-1}\Big) \quad \text{for all } r_i \in R.$$

Remember $a_i^1 \in L_c \setminus Z_{c-1}(L_c)$, $a_i^{j-1} \in Z_{c-2-j}(L_c) \setminus Z_{c+1-j}(L_c)$. Let $jn < k$. It suffices to prove

(3) $\qquad \neg o_{j,n}(a) \quad \text{for } a = \sum_{i<k} (a_i^1, a_i^{j-1}).$

Assume $a \in \langle B \rangle$ modulo $Z_{c-j}(L_c)$ where B is an $(o)_{j,n}$-system. Then we can extend $\{a_0^1, \ldots, a_{k-1}^1\}$ to an (o)-system

$$C = \{c_{\alpha(i)}^i : 1 \leq i < j, \alpha(1) < k + n, \alpha(i) < in \text{ for } 1 < i < j\}$$

such that $c_{\alpha(1)}^1 = a_{\alpha(1)}^1$ for $\alpha(1) < k$ and $B \subseteq \langle C \rangle$ modulo Z_{c-j}. Since

$$\neg o_{j-1,jn+k}\Big(\sum_{i<k} r_i a_i^{j-1}\Big) \quad \text{for all } r_i \in R$$

and since $jn < k$ there are some $a_{i_1}^{j-1}, \ldots, a_{i_s}^{j-1}$ such that $D = C \cup \{a_{i_1}^{j-1}, \ldots, a_{i_s}^{j-1}\}$ is an (o)-system with $A \subseteq D$ modulo Z_{c-j}.

Then a has two different presentations as linear combinations of basic monomials on D modulo Z_{c-j}. This is a contradiction by Corollary 2.7. □

3. Subgroups of $F_c(p, \kappa)$

The investigation of free nilpotent groups is closely related to that of free Lie algebras. The main results are based on the work of P. Hall [12, 13], Magnus [14], Witt [23], and M. Hall [10]. As pointed out by P. Hall there are similar results for $F_c(p^n, \kappa)$. Using the Theorem of Širšov–Witt (Theorem 2.4) we characterize the subgroups of $F_c(p, \kappa)$.

We use Z to denote the integers. If G is an arbitrary group and $x, y \in G$ let $[x, y]$ be the element $x^{-1}y^{-1}xy$ as usual. If A is a subset of G we denote by $\langle A \rangle$ the subgroup generated by A. The elements $\Gamma_n(G)$ of the lower central series of G, the elements $Z_n(G)$ of the upper central series of G, nilpotency of class $\leqslant c$ of G, and *basic commutators of A-weight n* on a subset A of G are defined as the corresponding notions for Lie algebras replacing (,) by [,] (commutator = monomial).

Theorem 3.1 (Second Basis theorem of P. Hall). *Let $A = \{a_\alpha : \alpha < \kappa\}$ be a set of free generators of $F_c(p^n, \kappa)$. Then for every $i \leqslant c$ $\Gamma_i(F_c(p^n, \kappa))/\Gamma_{i+1}(F_c(p^n, \kappa))$ is a free module over the ring Z/p^nZ. The cosets of all basic commutators of A-weight i form a basis of that module.*

Corollary 3.2. *Assume $\{b_\alpha : \alpha < \lambda\}$ is a sequence of all basic commutators on A (of A-weight less then $c+1$) ordered according to A-weight. Then every element g of $F_c(p^n, \kappa)$ can be uniquely expressed as $g = \prod_\alpha b_\alpha^{r_\alpha}$ where $r_\alpha \in Z/p^nZ$ and $r_\alpha = 0$ up to finitely many α.*

Theorem 3.1 is an unpublished result of P. Hall. In [9] Gruenberg proved it for nilpotent products of cyclic groups of given order p^n. Afterwards it was generalized by Struik [22]. There is a well-known procedure to construct for a given group a corresponding Lie algebra over Z. In the case of $F_c(p, \kappa)$ we get a Lie algebra $L[F_c(p, \kappa)]$ (short $L[F_c]$) over the field Z/pZ isomorphic to $L_c(Z/pZ, \kappa)$. As a Z/pZ-module let $L[F_c]$ be $\bigoplus_{1\leqslant i\leqslant c} L[F_c]^i$ where $L[F_c]^i = \Gamma_i(F_c)/\Gamma_{i+1}(F_c)$. For $\bar{a} = \sum_{1\leqslant i\leqslant c} \bar{a}_i$ and $\bar{b} = \sum_{1\leqslant i\leqslant c} \bar{b}_i$ where $\bar{a}_i, \bar{b}_i \in L[F_c]^i$ choose elements $a_i, b_i \in F_c$ in the cosets $\bar{a}_i$ resp. $\bar{b}_i$. Then $(\bar{a}, \bar{b}) = \sum_{1\leqslant i\leqslant c} \bar{c}_i$, where $\bar{c}_i \in L[F_c]^i$ and $\bar{c}_i$ is the coset of $\prod_{r+s=i} [a_r, b_s]$. $(\bar{a}, \bar{b})$ is well defined, and it is a Lie product. To prove this use Theorem 5.3 in [15, p. 293].

Theorem 3.3. *Let $\{a_\alpha : \alpha < \kappa\}$ be a set of free generators of $F_c(p, \kappa)$, and $\{c_\alpha : \alpha < \kappa\}$ be a set of free generators of $L_c(Z/pZ, \kappa)$. Then $\varphi(c_\alpha) = a_\alpha \Gamma_2(F_c)$' induces an isomorphism of $L_c(Z/pZ, \kappa)$ onto $L[F_c(p, \kappa)]$.*

Proof. Assume that the basic commutators on $A = \{a_\alpha : \alpha < \kappa\}$ are constructed by the same procedure as the basic monomials on $C = \{c_\alpha : \alpha < \kappa\}$. By Theorem 3.1 $L[F_c]^i$ is a vector space over Z/pZ, and the cosets of the basic commutators of C-weight n form a basis of this space. Therefore and by Lemma 2.2 the canonical homomorphism φ from $L_c(Z/pZ, \kappa)$ onto $L[F_c]$ defined by $\varphi(c_\alpha) = a_\alpha \Gamma_2(F_c)$ is an isomorphism. By Theorem 3.3 $L_c(Z/pZ, \kappa)$ is interpreted in $F_c(p, \kappa)$. □

Corollary 3.4. *$\Gamma_i(F_c(p^n, \kappa)) = Z_{c+1-i}(F_c(p^n, \kappa))$ for $1 \leqslant i \leqslant c+1$.*

Proof. It is clear that $\Gamma_i \subseteq Z_{c+1-i}$, in particular $\Gamma_c \subseteq Z_1$. Using induction on c it is sufficient to show that $\Gamma_c = Z_1$. At first we consider the case $n = 1$. By the theorem above the assertion follows from the corresponding fact for $L_c(Z/pZ, \kappa)$. For the general case assume that $a \notin \Gamma_c$. Then $a \in \Gamma_m \setminus \Gamma_{m+1}$ for some $m < c$. By Theorem 3.1 $a = (\prod_\alpha b_\alpha^{r_\alpha})^{p^s} \bmod \Gamma_{m+1}$, where the b_α's are basic commutators of A-weight m,

$s<n$, and w.l.o.g. $p \nmid r_1$. Let $d=\prod_\alpha b_\alpha^{r_\alpha}$. W.l.o.g. we can further suppose $a=d^{p^s}$. There is a natural homomorphism φ of $F_c(p^n, \kappa)$ onto $F_c(p, \kappa)$, that maps the set A of generators of $F_c(p^n, \kappa)$ onto the set $\bar{A}$ of generators of $F_c(p, \kappa)$. Since $p \nmid r_1$ $\varphi(d)=\prod_\alpha \varphi(b_\alpha)^{r_\alpha} \notin \Gamma_{m+1}(F_c(p, \kappa))$ by Theorem 3.1 for the case $n=1$. Since the assertion is true for $F_c(p, \kappa)$ there is some $\varphi(e)$ such that $[\varphi(d), \varphi(e)] \notin \Gamma_{m+2}(F_c(p, \kappa))$. Therefore $[d, e]$ is not a p-power in $\Gamma_{m+1}(F_c(p^n, \kappa))/\Gamma_{m+2}(F_c(p^n, \kappa))$. But then $[d, e]^{p^s} \notin \Gamma_{m+2}(F_c(p^n, \kappa))$. Since $[d, e]^{p^s}=[a, e] \bmod \Gamma_{m+2}(F_c(p^n, \kappa))$ and $m+2 \leq c+1$, $a \notin Z_1(F_c(p^n, \kappa))$. □

Theorem 3.3 allows us to apply the Theorem of Širšov–Witt (Theorem 2.4). In our translation of the notions of (o)- and $(*)$-systems in group theory we use again the upper central series since $Z_0, Z_1, \ldots$ are definable in the elementary language. Let G be a cth nilpotent group with the following property:

(Z) $\quad [Z_{c+1-i}(G), Z_{c+1-j}(G)] \subseteq Z_{c+1-(i+j)}(G) \quad$ for $1 \leq i, j \leq c+1$.

Let $A=\{a^i_{\alpha(i)}: 1 \leq i \leq c, \alpha(i)<\lambda_i\}$ be a subset of G such that $a^i_{\alpha(i)} \in Z_{c+1-i}(G)$. We fix the following order on A: $a^i_\alpha < a^j_\beta$ iff $i<j$ or $i=j$ and $\alpha<\beta$. As for Lie algebras define the *A-degree* $d_A(b)$ of a commutator b: $d_A(a^i_\alpha)=i$ and $d_A([b_1, b_2])=d_A(b_1)+d_A(b_2)$.

A is an *(o)-system* iff for every $n \leq c$ the elements of $\{a^n_{\alpha(n)}: \alpha(n)<\lambda_n\}$ are linearly independent modulo the normal subgroup generated by $Z_{c-n}(G)$ and all basic commutators on $\{a^i_{\alpha(i)} \in A: i<n\}$ of A-degree n.

A is a *(*)-system* iff for every $n \leq c$ the basic commutators of A-degree n are linearly independent modulo $Z_{c-n}(G)$.

Lemma 3.5. *Let G be a cth nilpotent group of exponent p with (Z).*

(i) *For every subgroup of G there exists a generating (o)-system.*

(ii) *Every (*)-system is an (o)-system.*

(iii) *A is an (o)-system ((*)-system) in G iff every finite subset of A is an (o)-system ((*)-system) in G.*

(iv) *$A=\{a^i_{\alpha(i)}: 1 \leq i \leq c, \alpha(i)<\lambda_i\}$ is an (o)-system in $F_c(p, \kappa)$ iff for every n $\{a^n_{\alpha(n)} \in A: \alpha(n)<\lambda_n\}$ is linearly independent modulo the normal subgroup generated by $Z_{c-n}(F_c)$ and $Z_{c+1-n}(F_c) \cap \langle\{a^j_{\alpha(j)} \in A: j<n\}\rangle$.*

(iv) is a consequence of the following Theorem 3.6 and Lemma 3.7.

Theorem 3.6. *Every (o)-system is a (*)-system in $F_c(p, \kappa)$.*

Proof. By Corollary 3.4 $Z_{c+1-i}(F_c(p, \kappa))=\Gamma_i(F_c(p, \kappa))$. Using Theorem 3.3 the assertion follows from the corresponding result concerning $F_c(Z/pZ, \lambda)$ (Theorem 2.4). □

Lemma 3.7. *Let $A=\{a^i_{\alpha(i)}: 1 \leq i \leq c, \alpha(i)<\lambda_i\}$ be a (*)-system of a cth nilpotent* *group G of exponent p with the property (Z). Let $\{b_\alpha : \alpha < \kappa\}$ be any enumeration of all basic commutators on A ordered according to A-degree. Then every element g of $\langle A\rangle$ can be uniquely expressed as $g = \prod_\alpha b_\alpha^{r_\alpha}$, where $r_\alpha \in Z/pZ$ and $r_\alpha = 0$ up to finitely many α. If $g \in Z_{c+1-n}(G) \cap \langle A\rangle$, then $r_\alpha = 0$ for all α such that b_α has A-degree $<n$.*

Proof. We prove this similar as Lemma 2.5. Let $\{b_\alpha : \alpha < \kappa_n\}$ be the set of all basic commutators b of A-degree $\leq n$. First of all we show:

(1) $\{b_\alpha : \kappa_n \leq \alpha < \kappa\}$ generates $Z_{c-n}(G) \cap \langle A\rangle$.

Let g be any element of $Z_{c-n}(G) \cap \langle A\rangle$. Let $\{c_\alpha : \alpha < \kappa\}$ be an enumeration of the b_α's according to their definition as basic commutators on A. As well known (see e.g. [11]) $g = \prod_\alpha c_\alpha^{s_\alpha}$, where $s_\alpha \in Z/pZ$ and $s_\alpha = 0$ up to finitely many α. Let $c_{\alpha_1}, \ldots, c_{\alpha_m}$ be the basic commutators in this presentation of g of minimal A-degree t such that $s_{\alpha_i} \neq 0$. Since A is a (*)-system, the c_{α_i}'s are linearly independent modulo $Z_{c-t}(G)$ and therefore $\prod_{1\leq i\leq m} c_{\alpha_i}^{s_{\alpha_i}} \neq 1$ modulo $Z_{c-t}(G)$. Hence $g \neq 1$ modulo $Z_{c-t}(G)$, and therefore $t > n$. This implies (1).

Now it is easy to prove the assertion by induction on c. Apply the induction hypothesis to $G/Z_1(G)$. □

There are some consequences of Theorem 3.6 and Lemma 3.7 similar as for Lie algebras.

Corollary 3.8. *If A is an (o)-system of $F_c(p, \kappa)$ and $\{b_\alpha : \alpha < \lambda\}$ is an enumeration of the basic commutators on A ordered according to A-degree, then every element g of $\langle A\rangle$ can be uniquely expressed as $g = \prod_\alpha b_\alpha^{r_\alpha}$, where $r_\alpha \in Z/pZ$ and $r_\alpha = 0$ up to finitely many α.*

Corollary 3.9. *Let A be a set of elements of $\Gamma_n(F_c(p, \kappa))$ linearly independent modulo $\Gamma_{n+1}(F_c(p, \kappa))$. Then A freely generates a subgroup isomorphic to $F_{[c/n]}(p, \kappa)$, where $[c/n]$ denotes the integral part of c/n.*

Corollary 3.10. *If $A = \{a^i_{\alpha(i)} : 1 \leq i \leq c, \alpha(i) < \lambda_i\}$ and $C = \{c^i_{\alpha(i)} : 1 \leq i \leq c, \alpha(i) < \lambda_i\}$ are two (o)-systems in $F_c(p, \kappa)$, then $\varphi(a^i_{\alpha(i)}) = c^i_{\alpha(i)}$ induces an isomorphism of $\langle A\rangle$ onto $\langle C\rangle$.*

Proof. There is an uniform procedure to bring a word w over A resp. C into the form $\prod_\alpha b_\alpha^{r_\alpha}$, where the b_α are the basic commutators ordered according to their definition. Using $uv = vu[uv]$ we get $w = b_{\alpha_1}^{r_{\alpha_1}} \cdots b_{\alpha_m}^{r_{\alpha_m}} w'$ and $\varphi(w) = \varphi(b_{\alpha_1})^{r_{\alpha_1}} \cdots \varphi(b_{\alpha_m})^{r_{\alpha_m}} \varphi(w')$ such that the b_{α_i}'s and $\varphi(b_{\alpha_i})$'s are of minimal A-degree n and $w', \varphi(w') \in Z_{c-n}(F_c)$. By Theorem 3.6 A and C are (*)-systems and therefore $w \neq 1$ iff $\varphi(w) \neq 1$. It follows that φ is well defined and is an isomorphism. □

Lemma 3.11. *Let H be a subgroup of $F_c(p,\kappa)$ and $A=\{a^i_{\alpha(i)}: 1\leq i\leq c, \alpha(i)<\lambda_i\}$ and $C=\{c^i_{\alpha(i)}: 1\leq i\leq c, \alpha(i)<\kappa_i\}$ be two generating (o)-systems of H. Then $\lambda_i=\kappa_i$ for $1\leq i\leq c$.*

This lemma is similarly proved as Lemma 2.10. By Lemma 3.5(i), Corollary 3.10, and Lemma 3.11 we obtain:

Theorem 3.12. *The subgroups of $F_c(p,\kappa)$ are characterized up to isomorphisms by the parameter λ_i $(1\leq i\leq c)$ of any generating (o)-system $A=\{a^i_{\alpha(i)}: 1\leq i\leq c, \alpha(i)<\lambda_i\}$ of H.*

We say that H is a $(\lambda_1,\ldots,\lambda_c)$-*subgroup* of $F_c(p,\kappa)$. This generalizes a subgroup theorem of Goldina [8] for the case $c=2$.

Corollary 3.13. *Let $A=\{a^i_{\alpha(i)}: 1\leq i\leq c, \alpha(i)<\lambda_i\}$ be a $(*)$-system of a cth nilpotent group G of exponent p with the property (Z). Then A is isomorphic to every $(\lambda_1,\ldots,\lambda_c)$-subgroup of every $F_c(p,\kappa)$ with $\lambda_i\leq\kappa$.*

Theorem 3.14. Let *H be a $(\lambda_1,\ldots,\lambda_c)$-subgroup of $F_c(p,\kappa)$ such that $2\leq\lambda_1$. Then $Z_i(F_c(p,\kappa))\cap H=Z_i(H)$. Therefore $A\subseteq H$ is an (o)-system ($(*)$-system) in H iff A is an (o)-system ($(*)$-system) in $F_c(p,\kappa)$.*

Proof. The same idea as for Lie algebras works. □

4. The elementary theory of $F_c(p,\kappa)$

We consider $F_c(p,\kappa)$ in an elementary language $\mathfrak{L}$ with one symbol for the group multiplication and a constant for '1'. For every c-tuple $(\lambda_1,\ldots,\lambda_c)$ of natural numbers there are elementary formulas $\varphi_{(o)}(x^1_0,\ldots,x^1_{\lambda_1-1},\ldots,x^c_0,\ldots,x^c_{\lambda_c-1})$ and $\varphi_{(*)}(x^1_0,\ldots,x^1_{\lambda_1-1},\ldots,x^c_0,\ldots,x^c_{\lambda_1-1})$ such that for every subset $A=\{a^i_{\alpha(i)}: 1\leq i\leq c, \alpha(i)<\lambda_i\}$ of a cth nilpotent group G of exponent p with the property (Z) the following hold:

$$G\models\varphi_{(o)}(a^1_0,\ldots,a^1_{\lambda_1-1},\ldots,a^c_0,\ldots,a^c_{\lambda_c-1}) \quad \text{iff } A \text{ is an } (o)\text{-system in } G,$$
$$G\models\varphi_{(*)}(a^1_0,\ldots,a^1_{\lambda_1-1},\ldots,a^c_0,\ldots,a^c_{\lambda_c-1}) \quad \text{iff } A \text{ is a } (*)\text{-system in } G.$$

It is possible to find $\varphi_{(o)}$ and $\varphi_{(*)}$ because the members of the upper central series are definable.

Therefore there is a recursive set $\Sigma_c(p)$ of elementary sentences expressing the following properties of a group G:

($\Sigma1$) G is a cth nilpotent group of exponent p with the property (Z).
($\Sigma2$) $G/Z_{c-1}(G)$ is infinite.
($\Sigma3$) Every finite (o)-system of G is a $(*)$-system.

Theorem 4.1. *G is a model of $\Sigma_c(p)$ iff G is isomorphic to a $(\lambda_1, \ldots, \lambda_c)$-subgroup of some $F_c(p, \kappa)$ such that $\kappa \geq \lambda_1 \geq \omega$. Particularly $F_c(p, \kappa)$ is a model of $\Sigma_c(p)$ iff $\kappa \geq \omega$.*

Proof. At first we prove that every $(\lambda_1, \ldots, \lambda_c)$-subgroup G of some $F_c(p, \kappa)$ such that $\kappa \geq \lambda_1 \geq \omega$ models $\Sigma_c(p)$. Use Theorem 3.14. Then $(\Sigma 3)$ is Theorem 3.6.

For the other direction assume G is a model of $\Sigma_c(p)$. By $(\Sigma 1)$ G is a cth nilpotent group of exponent p with (Z). By Lemma 3.5(i) there exists an (o)-system $A = \{a^i_{\alpha(i)}: 1 \leq i \leq c, \alpha(i) < \lambda_i\}$ generating G. By $(\Sigma 2)$ $\lambda_1 \geq \omega$. By $(\Sigma 3)$ and Lemma 3.5(iii) A is a $(*)$-system. Then Corollary 3.13 implies that G is isomorphic to a $(\lambda_1, \ldots, \lambda_c)$-subgroup of some $F_c(p, \kappa)$, where $\kappa \geq \lambda_1 \geq \omega$. □

Theorem 4.2. *The elementary theory of every model of $\Sigma_c(p)$ is ω-stable.*

Proof. Let G be any countable model of $\Sigma_c(p)$. We will show that there are countably many 1-types over G only. There is an elementary extension of G such that every 1-type $p(x)$ over G is realized in H. Then H is a model of $\Sigma_c(p)$ and therefore H is a $(\lambda_1, \ldots, \lambda_c)$-subgroup of some $F_c(p, \kappa)$ such that $\lambda_1 \geq \omega$ (Theorem 4.1). Let $A = \{a^i_{\alpha(i)}: 1 \leq i \leq c, \alpha(i) < \lambda_i\}$ be a generating (o)-system of H in $F_c(p, \kappa)$.

Denote $A_1 = \{a^i_{\alpha(i)} \in A: \alpha(i) < \omega\}$ and $A_2 = \{a^i_{\alpha(i)} \in A: \alpha(i) < \omega + \omega\}$. We can order A in such a way that $G \subseteq \langle A_1 \rangle$ (Lemma 3.5(iv)). It is sufficient to show that for every $a \in H$ there is some $b \in \langle A_2 \rangle$ and an automorphism φ of H such that φ is the identity on $\langle A_1 \rangle$ and $\varphi(a) = b$.

Let $C = \{c^i_{\alpha(i)}: 1 \leq i \leq c, \alpha(i) < n\}$ be a subset of $A \setminus A_1$ such that $a \in \langle A_1 \cup C \rangle$. We define a permutation $\mathfrak{S}$ of A as follows: $\mathfrak{S}(a^i_{\alpha(i)}) = a^i_{\alpha(i)}$ if $a^i_{\alpha(i)} \neq c^i_j$ and $a^i_{\alpha(i)} \neq a^i_{\omega+j}$ for every j. $\mathfrak{S}(c^i_j) = a^i_{\omega+j}$ and $\mathfrak{S}(a^i_{\omega+j}) = c^i_j$.

By Lemma 3.5(iv) $\mathfrak{S}(A)$ is again an (o)-system in $F_c(p, \kappa)$. By Corollary 3.10. $\mathfrak{S}$ induces the desired automorphism. □

Corollary 4.3. *If G models $\Sigma_c(p)$ and* card$(G) \leq \kappa$, *then there exists an elementary saturated extension of G of cardinality κ.*

Let $A = \{a^i_{\alpha(i)}: 1 \leq i \leq c, \alpha(i) < \lambda_i\}$ be an (o)-system in a subgroup G of some $F_c(p, \kappa)$. If $\lambda_i \leq n$ and $\lambda_i = 0$ for $i \geq j$ we call A an *$(o)_{j,n}$-system*, similarly as for Lie algebras. Furthermore we define $G \models o_{j,n}(a)$ iff $a \in Z_{c+1-j}(G)$ and there is an $(o)_{j,n}$-system A such that a is an element of $\langle A \rangle$ modulo $Z_{c-j}(G)$. For $n = 0$ $G \models o_{j,0}(a)$ iff $a \in Z_{c-j}(G)$. The predicates $o_{j,n}$ are elementary definable, since every element of $\langle A \rangle$, where A is an $(o)_{j,n}$-system, has a presentation $\prod_\alpha b_\alpha^{r_\alpha}$ such that $r_\alpha \in Z/pZ$ and the b_α's are basic commutators on A of A-degree $\leq c$.

Lemma 4.4. *Let G be a $(\lambda_1, \ldots, \lambda_c)$-subgroup of some $F_c(p, \kappa)$ with $\lambda_1 \geq \omega$. For every $1 < j \leq c$, every $n < \omega$, and every given elements $c_1, \ldots, c_m$ of $Z_{c+1-j}(G)$ there* *is some* $a \in Z_{c+1-j}(G)$ *such that*

$$G \models \neg o_{j,n}\Big(a^{-1} \prod_{1 \leq i \leq m} c_i^{r_i}\Big) \quad \textit{for all } r_i \in Z/pZ.$$

Proof. The lemma follows from the corresponding Lemma 2.15 for $L_c(Z/pZ, \kappa) \cong L[F_c(p, \kappa)]$. $\square$

Corollary 4.5. *Let* G *be a saturated model of* $\Sigma_c(p)$ *of cardinality* λ. *Then* G *is a* $(\lambda, \ldots, \lambda)$-*subgroup of every* $F_c(p, \kappa)$ *such that* $\kappa \geq \lambda$.

Proof. By Theorem 4.1 G is a $(\lambda_1, \ldots, \lambda_c)$-subgroup of some $F_c(p, \kappa)$ and by Corollary 3.13 of every $F_c(p, \kappa)$ where $\kappa \geq \lambda_i$ and $\lambda_1 \geq \omega$. We prove $\lambda_j = \lambda$ by induction on j. $\lambda_1 = \lambda$ follows from ($\Sigma 2$). Assume $\lambda_j < \lambda$. We consider the following set $p(x)$ of sentences:

$$\{o_{j-1,0}(x)\} \cup \Big\{\neg o_{j,n}\Big(x^{-1} \prod_i c_i^{r_i}\Big) : r_i \in Z/pZ,\ c_i \in \{a^j_{\alpha(j)} \in A : \alpha(j) < \lambda_j\},\quad n < \omega\Big\}$$

where

$$A = \{a^i_{\alpha(i)} : 1 \leq i \leq c,\ \alpha(i) < \lambda_i\}$$

is a generating (o)-system of G.

By Lemma 4.4 $p(x)$ is a consistent type over G with less than λ-many constants from G. Since G is saturated $p(x)$ is realized by some $a \in Z_{c+1-j}(G)$. By the definition of $p(x)$ $a \notin \langle\{a^i_{\alpha(i)} \in A : i \leq j\}\rangle \supseteq Z_{c+1-j}(G)$, a contradiction. $\square$

Theorem 4.6. $\Sigma_c(p)$ *is complete.*

Proof. Let G and H be two countable models of $\Sigma_c(p)$. By Corollary 4.3 we can assume that G and H are saturated. By Corollary 4.5 G and H are $(\omega, \ldots, \omega)$-subgroups of some $F_c(p, \kappa)$ and therefore isomorphic by Theorem 3.13. $\square$

Corollary 4.7. *For every* $\kappa \geq \omega$ $F_c(p, \kappa) \equiv F_c(p, \omega)$. *Also the theory* $\mathrm{Th}(F_c(p, \omega))$ *axiomatized by* $\Sigma_c(p)$ *is decidable.*

We obtain the language $\mathfrak{L}^*$ from $\mathfrak{L}$ adding new predicates $o_{j,n}(x)$ for all $1 < j \leq c$, $0 \leq n < \omega$ *and* $j = 1$, $n = 0$. Let $\Sigma_c^*(p)$ be the extension of $\Sigma_c(p)$ in $\mathfrak{L}^*$ with the definitions for the new predicates $o_{j,n}(x)$. We write $H \overset{*}{\subseteq} G$ iff H is a substructure of G with respect to $\mathfrak{L}^*$.

Theorem 4.8. $\Sigma_c^*(p)$ *admits the elimination of quantifiers.*

Proof. As well known the assertion is equivalent to the following (see e.g. [19]):

Every diagram of the following sort can be completed as shown:

(1) $$\begin{tikzcd} & K & \\ M \arrow[ur, dashed, "\cong"] & & N \arrow[ul, dashed, "*"'] \\ & H \arrow[ul, "*"] \arrow[ur, "*"'] & \end{tikzcd} \qquad M, N \models \Sigma_c^*(p).$$

Assume M, H, N are given as in (1) above. Let $\lambda = \max(\{\mathrm{card}(M), \mathrm{card}(N)\})$. By Corollary 4.3 we can assume that M and N are saturated models of $\Sigma_c(p)$ of cardinality λ. Using Corollary 4.5 we can further suppose that M and N are $(\lambda, \ldots, \lambda)$-subgroups of $F_c(p, \kappa)$. Then (1) follows from Corollary 3.10, if we have shown:

(2) If $H \overset{*}{\subseteq} G \subseteq F_c(p, \kappa)$, then it is possible to extend every generating (o)-system of H to a generating (o)-system of G.

Let A be $\{a^i_{\alpha(i)} : 1 \leq i \leq c, \alpha(i) < \lambda_i\}$. Since $H \overset{*}{\subseteq} G$, A is an (o)-system in G. By induction on j we construct (o)-systems B_j that contain $\{a^i_{\alpha(i)} \in A : i \leq j\}$ and generate G modulo $Z_{c-j}(G)$. Then B_c fulfils (2).

Assume $B_{j-1} = \{b^i_{\alpha(i)} : 1 \leq i \leq j-1, \alpha(i) < \kappa_i\}$ is constructed and has the desired properties. Then it suffices to prove that $C = B_{j-1} \cup \{a^j_{\alpha(j)} \in A : \alpha(j) < \lambda_j\}$ is an (o)-system of G. If this would be false we would get

$$G \models Z_{c-j}\Big(\prod_{k<n} d_k^{r_k} \prod_{l<m} a_l^{s_l}\Big),$$

where $r_k, s_l \in Z/pZ$, $s_l \neq 0$ for some l, $a_l \in \{a^j_{\alpha(j)} \in A : \alpha(j) < \lambda_j\}$, and the d_k's are basic commutators on B_{j-1} of C-degree j. This means there is some h such that $G \models o_{j,h}(\prod_{l<m} a_l^{s_l})$. Hence by $H \overset{*}{\subseteq} G$ $H \models o_{j,h}(\prod_{l<m} a_l^{s_l})$, a contradiction. $\square$

Remark 4.9. $\mathrm{Th}(\{F_c(p, n) : n \text{ natural } > 1\})$ is decidable and has the following axiomatization:

$$(\Sigma_c(p) \setminus (\Sigma 2)) \cup (\text{`card}(G/Z_{c-1}(G)) > p\text{'})$$
$$\cup(\{\text{`If card}(G/Z_{c-1}(G)) = p^n\text{, then } G \cong F_c(p, n)\text{'} : n > 1, \text{natural}\}).$$

5. Decidability and ω-stability of $F_c(p^n, \kappa)$

In this section we extend our results concerning $F_c(p, \kappa)$ to free cth nilpotent p-groups of finite exponent $(c < p)$. Following P. Hall [13] we define for every p-group G $\mho_k(G) = \langle\{g^{p^k} : g \in G\}\rangle$.

Theorem 5.1 (Hall [13]). *Every p-group G of nilpotency class less than p is regular. Hence for each k* $\mho_k(G)$ *is the set of all* g^{p^k} *for* $g \in G$.

For convenience write $\mho(G)$ instead of $\mho_1(G)$. By the theorem above $\mho_k(G)$ is elementarily definable. Therefore there is a recursive set $\Sigma_c(p^n)$ of elementary sentences expressing the following properties of a group G:

(1) G is a cth nilpotent group of exponent p^n satisfying (Z).

(2) $G/\mho(G)$ is a model of $\Sigma_c(p)$.

(3) Every coset of $Z_i(G/\mho(G))$ contains some element of $Z_i(G)$ $(1 \leq i \leq c)$.

(4) $Z_i(G)/Z_{i-1}(G)$ is isomorphic to a direct sum of copies of the cyclic group of order p^n $(1 \leq i \leq c)$.

The following formulas describe (4). For $1 \leq k < n$ consider $\forall x(Z_i(x) \wedge Z_{i-1}(x^{p^k}) \wedge \neg Z_{i-1}(x^{p^{k-1}}) \rightarrow \exists y\, \exists z\ (Z_i(y) \wedge Z_{i-1}(z) \wedge x = y^{p^{n-k}} + z))$.

Lemma 5.2. *For $\omega \leq \kappa$, $F_c(p^n, \kappa)$ is a model of $\Sigma_c(p^n)$.*

Proof. Axiom (1) is satisfied by definition and Corollary 3.4. Since $F_c(p^n, \kappa)/\mho(F_c(p^n, \kappa))$ is a free object in the variety of all cth nilpotent groups of exponent p it is isomorphic to $F_c(p, \kappa)$. This implies Axiom (2). Let φ be a homomorphism that maps a set C of free generators of $F_c(p^n, \kappa)$ onto a set $\bar{C}$ of free generators of $F_c(p, \kappa)$. Assume $\varphi(a) \in Z_i(F_c(p, \kappa))$ and let $a = \prod_\alpha b_\alpha^{r_\alpha} \prod_\alpha d_\alpha^{s_\alpha}$ be the presentation of a according to P. Hall's Second Basis theorem, where the b_α's are the basic commutators of C-weight $< i$ and the d_α's are the basic commutators of C-weight $\geq i$. Since $\varphi(a) \in Z_i(F_c(p, \kappa))$, $p \mid r_\alpha$ for all r_α. Then $a_1 = \prod_\alpha d_\alpha^{s_\alpha}$ is an element of $Z_i(F_c(p^n, \kappa))$ and $\varphi(a) = \varphi(a_1)$, as desired in Axiom (3). Axiom (4) is an immediate consequence of P. Hall's Second Basis theorem, if we have regard to $Z_{c+1-i}(F_c(p^n, \kappa)) = \Gamma_i(F_c(p^n, \kappa))$ (Corollary 3.4). □

Let G be a model of $\Sigma_c(p^n)$. For elements a of G and subsets A of G we use $\bar{a}$ resp. $\bar{A}$ to denote their images in $G/\mho(G)$. A subset $A = \{a^i_{\alpha(i)}: 1 \leq i \leq c, \alpha(i) < \lambda_i\}$ of G is called a (□)-system iff $\bar{A}$ is a (*)-system in $G/\mho(G)$, $\bar{A}$ generates $G/\mho(G)$, and $a^i_{\alpha(i)} \in Z_{c+1-i}(G)$.

Lemma 5.3. *Let G be a model of $\Sigma_c(p^n)$.*

(i) *There is a (□)-system in G.*

(ii) *If A is a (□)-system in G, then the basic commutators on A of A-degree i form a basis of the free module $Z_{c+1-i}(G)/Z_{c-i}(G)$.*

(iii) *Let A be a (□)-system in G. Let $\{b_\alpha: \alpha < \kappa\}$ by any enumeration of all basic commutators on A ordered according to A-degree. Then every element g of G can be uniquely expressed as $g = \prod_\alpha b_\alpha^{r_\alpha}$, where $r_\alpha \in Z/p^nZ$ and $r_\alpha = 0$ up to finitely many α.*

Proof. Ad (i): Let $\bar{A} = \{a^i_{\alpha(i)}: 1 \leq i \leq c, \alpha(i) < \lambda_i\}$ be a generating (o)-system of $G/\mho(G)$ (exists by Lemma 3.5(i)). Since $G/\mho(G)$ is a model of $\Sigma_c(p)$ (Axiom 2), $\bar{A}$ is a (*)-system of $G/\mho(G)$. By Axiom (3) there is an element $a^i_{\alpha(i)} \in Z_{c+1-i}(G)$ in every coset $\bar{a}^i_{\alpha(i)}$. Then $A = \{a^i_{\alpha(i)}: 1 \leq i \leq c, \alpha(i) < \lambda_i\}$ is a (□)-system of G.

(iii) follows immediately from (ii).

Ad (ii): Let $\{b_\alpha : \alpha < \kappa\}$ be a sequence of basic commutators on A and let $\{\bar{b}_\alpha : \alpha < \kappa\}$ be the sequence of the images of the b_α's in $G/\mho(G)$. Then every b_α of A-degree i is an element of $Z_{c+1-i}(G)$ by the property (Z). Notice that $a \in Z_i(G)$ implies $\bar{a} \in Z_i(G/\mho(G))$.

Let $\{b_\alpha : \alpha \in I\}$ be the set of basic commutators of A-degree i. At first we show that the b_α's with $\alpha \in I$ are linearly independent modulo $Z_{c-i}(G)$. Assume $\prod_{\alpha\in I_1} b_\alpha^{r_\alpha} \in Z_{c-i}(G)$ for a finite subset I_1 of I and $0 < r_\alpha < p^n$.

Then $\prod_{\alpha\in I_1} \bar{b}_\alpha^{r_\alpha} \in Z_{c-i}(G/\mho(G))$. $p \mid r_\alpha$ follows for $\alpha \in I_1$, since $\bar{A}$ is a (*)-system of $G/\mho(G)$. Let k be maximal with $p^k \mid r$ for all $\alpha \in I_1$. Then $k < n$, and $(\prod_{\alpha\in I_1} b_\alpha^{r_\alpha/p^k})^{p^k} \in Z_{c-i}(G)$. Now we choose $h \leq k$ minimal such that $(\prod_{\alpha\in I_1} b_\alpha^{r_\alpha/p^k})^{p^h} \in Z_{c-i}(G)$. By Axiom (4) and $h \leq k < n$, there is some c such that

$$c^{p^{n-h}} = \prod_{\alpha\in I_1} b_\alpha^{r_\alpha/p^k} \quad \text{modulo } Z_{c-i}(G).$$

But then

$$\prod_{\alpha\in I_1} \bar{b}_\alpha^{r_\alpha/p^k} \in Z_{c-i}(G/\mho(G)),$$

a contradiction.

It remains to prove that $\{b_\alpha : \alpha \in I\}$ generates $Z_{c+1-i}(G)$ modulo $Z_{c-i}(G)$. Let $a \in Z_{c+1-i}(G)$. Since $\bar{A}$ is a generating (*)-system of $G/\mho(G)$ it follows by Lemma 3.7

$$\bar{a} = \prod_{\alpha\in I_2} \bar{b}_\alpha^{r_\alpha} \quad \text{modulo } Z_{c-i}(G/\mho(G))$$

for some finite subset I_2 of I and $0 < r_\alpha < p$ for $\alpha \in I_2$. By Axiom (3) there is some d such that

$$a = \prod_{\alpha\in I_2} b_\alpha^{r_\alpha} d^p \quad \text{modulo } Z_{c-i}(G).$$

Let H be the image of $\langle\{b_\alpha : \alpha \in I\}\rangle$ in $Z_{c+1-i}(G)/Z_{c-i}(G)$. Then we have proved that $(Z_{c+1-i}(G)/Z_{c-i}(G))/H$ is divisible and therefore trivial. □

Corollary 5.4. *Let G and H be models of $\Sigma_c(p^n)$. Let $A = \{a^i_{\alpha(i)} : 1 \leq i \leq c, \alpha(i) < \lambda_i\}$ and $C = \{c^i_{\alpha(i)} : 1 \leq i \leq c, \alpha(i) < \kappa_i\}$ by (□)-systems of G resp. of H.*

(i) *If $\lambda_i = \kappa_i$ for $1 \leq i \leq c$, then there is an isomorphism φ of G onto H generated by $\varphi(a^i_{\alpha(i)}) = c^i_{\alpha(i)}$.*

(ii) *If $G = H$, then $\lambda_i = \kappa_i$ for $1 \leq i \leq c$ and $\lambda_1 \geq \omega$.*

(iii) *If $\mathfrak{S}_i$ is a permutation of the ordinals $\{\alpha : \alpha < \lambda_i\}$, then $\{a^i_{\mathfrak{S}_i}(\alpha(i)) : 1 \leq i \leq c, \alpha(i) < \lambda_i\}$ is a (□)-system.*

Proof. (ii) and (iii) follow from the corresponding properties of $\bar{A}$ and $\bar{C}$. (i) follows from Lemma 5.3(ii) as Corollary 3.10 from Theorem 3.6. The property (Z) is essentially used. □

By Corollary 5 4(i) and (ii), a model G of $\Sigma_c(p^n)$ is uniquely determined up to isomorphisms by the parameter $\lambda_1, \ldots, \lambda_c$ of any ($\square$)-system $\{a^i_{\alpha(i)}: 1 \leq i \leq c, \alpha(i) < \lambda_i\}$ in G. We say G is a $(\lambda_1, \ldots, \lambda_c)$-model. By the same proof as in Section 4 (Theorem 4.2), we get:

Theorem 5.5. *The elementary theory of every model of $\Sigma_c(p^n)$ especially of $F_c(p^n, \kappa)$ for $\omega \leq \kappa$ is ω-stable.*

Theorem 5.5 implies that for every model G of $\Sigma_c(p^n)$ and every $\kappa \geq \text{card}(G)$ there is an saturated elementary extension H of G of cardinality κ. By Corollary 4.5, H is a $(\kappa, \ldots, \kappa)$-model of $\Sigma_c(p^n)$. Therefore by Corollary 5.4(i) any two countable models of $\Sigma_c(p^n)$ have a common elementary extension. It follows:

Theorem 5.6. *$\Sigma_c(p^n)$ is complete. Therefore* $\text{Th}(F_c(p^n, \kappa))$ *is decidable and $F_c(p^n, \omega) \equiv F_c(p^n, \kappa)$ for every $\kappa \geq \omega$.*

6. Transfer theorems for free nilpotent Lie algebras over fields

Let R be a fixed field as in Section 2. In this section we consider the elementary theory of $L_c(R, \kappa)$ for $\kappa \geq \omega$. We use a two-sorted language $\mathfrak{L}_2$ with variables for elements of the field and variables for elements of the Lie algebra. $\mathfrak{L}_2$ contains symbols for addition and multiplication of the field and of the Lie algebra, a symbol for the action of the field on the Lie algebra, a constant for '0' of the Lie algebra, and constants for '0' and '1' of the field.

If we say 'H is a subalgebra of M', then this is meant in the sense of Section 2 where R is fixed. But we use H, M, N, K also to denote structures of $\mathfrak{L}_2$. In this case H, M, N, K are two-sorted structures. Let R_H, R_M, R_N, R_K be the corresponding fields. $H \subseteq M$ is the embedding with respect to $\mathfrak{L}_2$.

By Theorem 3.3 $L_c(Z/pZ, \kappa) \cong L[F_c(p, \kappa)]$. The construction of $L[F_c(p, \kappa)]$ from $F_c(p, \kappa)$ is an interpretation. Therefore we get

Corollary 6.1. *The elementary theory of $L_c(Z/pZ, \kappa)$ is decidable and ω-stable.*

Now we shall prove similar results considering arbitrary fields R. We use ideas that are developed in Section 4. For every c-tuple $(\lambda_1, \ldots, \lambda_c)$ of naturals there are elementary formulas $\varphi_{(o)}(x^1_0, \ldots, x^1_{\lambda_1-1}, \ldots, x^c_0, \ldots, x^c_{\lambda_c-1})$ and $\varphi_{(*)}(x^1_0, \ldots, x^1_{\lambda_1-1}, \ldots, x^c_0, \ldots x^c_{\lambda_c-1})$ such that for every subset $A = \{a^i_{\alpha(i)}: 1 \leq i \leq c, \alpha(i) < \lambda_i\}$ of a cth nilpotent Lie algebra M over a field the following hold:

$$M \models \varphi_{(o)}(a^1_0, \ldots, a^1_{\lambda_1-1}, \ldots, a^c_0, \ldots, a^c_{\lambda_c-1}) \quad \text{iff } A \text{ is an } (o)\text{-system.}$$

$$M \models \varphi_{(*)}(a^1_0, \ldots, a^1_{\lambda_1-1}, \ldots, a^c_0, \ldots, a^c_{\lambda_c-1}) \quad \text{iff } A \text{ is an } (*)\text{-system.}$$

Remember that the members Z_i of the upper central series are elementarily definable in the language $\mathfrak{L}_2$. Therefore we can define a set $\Delta_c(R)$ of formulas of $\mathfrak{L}_2$ expressing the following properties of a Lie algebra M over a field S:

(Δ0) $\mathrm{Th}(S) \equiv \mathrm{Th}(R)$.

(Δ1) M is a cth nilpotent Lie algebra over S satisfying (Z).

(Δ2) $M/Z_{c-1}(M)$ is a vector space over S of infinite dimension.

(Δ3) Every finite (o)-system of M is a (*)-system.

Remark 6.2. $\Delta_c(R)$ is recursive relative to $\mathrm{Th}(R)$.

Theorem 6.3. *M is a model of $\Delta_c(R)$ iff M is isomorphic to a $(\lambda_1, \ldots, \lambda_c)$-subalgebra of some $L_c(R_M, \kappa)$ such that $\kappa \geq \lambda_1 \geq \omega$ and $R \equiv R_M$. Particularly $L_c(R, \kappa)$ is a model of $\Delta_c(R)$ iff $\kappa \geq \omega$.*

Proof. At first we prove that every $(\lambda_1, \ldots, \lambda_c)$-subalgebra M of some $L_c(S, \kappa)$ with $\lambda_1 \geq \omega$ and $R \equiv S$ models $\Delta_c(R)$. (Δ1) and (Δ2) follow from $\Gamma_i(L_c(S, \kappa)) = Z_{c+1-i}(L_c(S, \kappa))$ and $Z_n(M) = Z_n(L_c(S, \kappa) \cap M$ (Lemma 2.13). Lemma 2.14 and Theorem 2.4 (Širšov–Witt) imply (Δ3).

For the other direction assume that M is a model of $\Delta_c(R)$. By (Δ0) and (Δ1) M is a cth nilpotent Lie algebra over a field S with (Z) and $S \equiv R$. By Lemma 2.3(i) there exists an (o)-system $A = \{a^i_{\alpha(i)}: 1 \leq i \leq c, \alpha(i) < \lambda_i\}$ generating M. By (Δ2) $\lambda_1 \geq \omega$. By (Δ3) and Corollary 2.3(iii) A is a (*)-system. Then Corollary 2.12 implies that M is isomorphic to a $(\lambda_1, \ldots, \lambda_c)$-subalgebra of some $L_c(S, \kappa)$ where $\lambda_1 \geq \omega$. □

To extend our language $\mathfrak{L}_2$ we introduce the following elementarily definable predicates $U_{j,n}(x_1, \ldots, x_m)$ for all $1 \leq j \leq c$, $m < \omega$, $n < \omega$:

$$M \models U_{j,n}(a_1, \ldots, a_m) \quad \text{iff } a_1, \ldots, a_m \in Z_{c+1-j}(M)$$

and there are an $(o)_{j,n}$-system $C \subseteq M$ (if $n = 0$, then $C = \emptyset$) and elements $r_1, \ldots, r_m \in R_M$ such that $\sum_{1 \leq i \leq m} r_i a_i$ is an element of the ideal generated by $Z_{c-j}(M)$ and the basic monomials of C-degree j on C.

We form a language $\mathfrak{L}_2^*$ from $\mathfrak{L}_2$ adding new predicate symbols $U_{j,n}(x_1, \ldots, x_m)$ for all $1 \leq j \leq c$, $m < \omega$, $n < \omega$, except if $j = 1$, then $n = 0$, and predicate symbols for the field theory such that the corresponding extension $\mathrm{Th}^*(R)$ by definitions of $\mathrm{Th}(R)$ admits elimination of quantifiers.

Let $\Delta_c^*(R)$ be the extension of $\Delta_c(R)$ in $\mathfrak{L}_2^*$ obtained by adding the defining axioms for the new predicates. $\mathfrak{L}_f^*$ is used to denote the sublanguage of $\mathfrak{L}_2$ concerning fields. Our aim is to prove that $\Delta_c^*(R)$ admits the elimination of quantifiers. We use $H \overset{*}{\leq} M$ to denote the fact that H is a substructure of M with respect to $\mathfrak{L}_2^*$. Corresponding to $H \overset{*}{\leq} M$ we write $R_H \overset{*}{\leq} R_M$. By construction of $\mathfrak{L}_2^*$ $R_H \overset{*}{\leq} R_M$ implies $R_H \leq R_M$.

Lemma 6.4. *Let M be a model of $\Delta_c(R)$ and H be a subalgebra of M ($\mathfrak{L}_2$-substructure with $R_H = R_M$). $A = \{a^i_{\alpha(i)}: 1 \le i \le c, \alpha(i) < \lambda_i\}$ be a generating (o)-system of H in H. Furthermore assume $\lambda_1 \ge 2$ and there is a generating (o)-system B of M extending A. Then $H \overset{*}{\subseteq} M$.*

Proof. We have to consider the predicates $U_{j,n}(x_1, \ldots, x_m)$ only. Let $\{b_\alpha: \alpha \le \lambda\}$ be an enumeration of the basic monomials on B. Since $M \models \Delta_c(R)$ by Theorem 6.3 there is some $L_c(R_M, \kappa)$ with $H \subseteq M \subseteq L_c(R_M, \kappa)$. Lemma 2.14 implies that A is a generating (o)-system of H in $L_c(R_M, \kappa)$ and B is a generating (o)-system of M in $L_c(R_M, \kappa)$ $(\lambda_1 \ge 2)$. Applying Lemma 2.13 in both cases we get $Z_i(M) \cap H = Z_i(H)$ for $1 \le i \le c$. Therefore $H \models U_{j,n}(a_1, \ldots, a_m)$ for $a_1, \ldots, a_m \in H$ implies $M \models U_{j,n}(a_1, \ldots, a_m)$. To prove the other direction suppose $M \models U_{j,n}(a_1, \ldots, a_n)$ for $a_1, \ldots, a_m \in H$. This means there is a nontrivial linear combination a of $a_1, \ldots, a_m$ and an $(o)_{j,n}$-system $C \subseteq M$ such that $a = \sum_\alpha r_\alpha d_\alpha$ modulo $Z_{c-j}(M)$ where $r_\alpha \in R_M$, and the d_α's are basic monomials on C of C-degree j.

B is a $(*)$-system of M, because M models $\Delta_c(R)$. Then from Lemma 2.5 follows:

(1) Every element c of M has a unique presentation $c = \sum_\alpha s_\alpha b_\alpha$, where $s_\alpha \in R_M$ and $s_\alpha = 0$ up to finitely many α. $c \in H$ iff all the b_α's with $s_\alpha \ne 0$ are basic monomials on A.

Let C be $\{c^i_{\alpha(i)}: 1 \le i \le j, \alpha(i) < k_i\}$ $(k_i \le n)$. By (1) $c^i_l = \sum_\gamma s^i_{l,\gamma} b_\gamma$ modulo $Z_{c-i}(M)$ where all b_γ's with $s^i_{l,\gamma} \ne 0$ have C-degree i. Then $c^i_l = e^i_l + \bar{e}^i_l$ where e^i_l is the sum of all '$s^i_{l,\gamma} b_\gamma$' with b_γ is a basic monomial on A, and $\bar{e}^i_l$ is the the sum of the other summands of c^i_l.

By the uniqueness in (1) and Lemma 2.2 every (b_α, b_β) is a linear combination of b_γ's that are not in H, if b_α or b_β is not in H. Therefore every basic monomial d_α on C has the form $d'_\alpha + \bar{d}_\alpha$ such that $d'_\alpha \in H$ is the corresponding basic monomial on the e^i_l's, and $\bar{d}_\alpha$ is a linear combination of b_γ's that are not in H. Since the presentation (1) is unique it follows $\sum_\alpha r_\alpha d_\alpha = \sum r_\alpha d'_\alpha$. Therefore we can replace C by $E = \{e^i_l: 1 \le i \le j, l < k_i\}$. For the relation $U_{j,n}(a_1, \ldots, a_m)$ the elements e^i_l modulo $Z_{c-i}(M)$ are essential only. It is easy to replace E by an $(o)_{j,n}$-system in H with the desired properties. □

Let M be a model of $\Delta_c(R)$, S be a subfield of R_M, and A be an (o)-system in M. Then we define $\langle A, S\rangle$ to be the substructure of M of all linear combinations of monomials on A with coefficients from S. $\langle A, S\rangle$ is a Lie algebra over S. By Corollary 2.7 and Theorem 6.3 every element of $\langle A\rangle$ can be uniquely expressed as $\sum_\alpha r_\alpha b_\alpha$ where the b_α's are basic monomials A and $r_\alpha \in R_M$ with $r_\alpha = 0$ up to finitely many α. If a is an element of $\langle A, S\rangle$, then the r_α's are elements of S. This follows since by Lemma 2.2 every monomial on A has the form $\sum t_\alpha b_\alpha$ where the b_α's are basic monomials on A and the t_α's are elements of the prime field of R_M. Hence $\langle A, S\rangle$ is a $(\lambda_1, \ldots, \lambda_c)$-subalgebra of some $L_c(S, \kappa)$ if $A = \{a^i_{\alpha(i)}: 1 \le i \le c, \alpha(i) < \lambda_i\}$. If $S = R_M$, then $\langle A, S\rangle = \langle A\rangle$.

Lemma 6.5. *Let M be a model of $\Delta_c(R)$, A be a generating (o)-system, and S be an elementary subfield of R_M. Then $\langle A, S\rangle \overset{*}{\preceq} M$ and $\langle A, S\rangle$ models $\Delta_c(R)$.*

Proof. Let $c_1, \ldots, c_m$ be any elements of $\langle A, S\rangle$ and $\varphi(x_1, \ldots, x_m)$ be an atomic formula of $\mathfrak{L}_2^*$. Let $A' = \{a^i_{\alpha(i)}: 1 \leq i \leq c, \alpha(i) < k_i\}$ be a finite subset of A with $k_1 \geq 2$ such that $c_1, \ldots, c_m \in \langle A', S\rangle$. By Lemma 6.4 you get $\langle A', R_M\rangle \overset{*}{\preceq} M$. Therefore it is sufficient to show

(2) $\quad \langle A', R_M\rangle \models \varphi(c_1, \ldots, c_m)$ iff $\langle A', S\rangle \models \varphi(c_1, \ldots, c_m)$.

Let $\{b_0, \ldots, b_s\}$ be the finite set of basic monomials on A'. By $(\Delta 3)$ A' is a $(*)$-system. By Lemma 2.5 every element of $\langle A', R_M\rangle$ can be uniquely expressed as $\sum_{0 \leq i \leq s} r_i b_i$. Using this we can interpret $\langle A', R_M\rangle$ in R_M coding the element $\sum_{0 \leq i \leq s} r_i b_i$ as the $(s+1)$-tuple $(r_0, \ldots, r_s)$. Multiplication of elements of $\langle A', R_M\rangle$ can be defined using Lemma 2.2. The other symbols of $\mathfrak{L}_2$ are defined component by component. Since $\mathfrak{L}_2^*$ is an extension by definitions the additional symbols of $\mathfrak{L}_2^*$ are defined automatically. The image of $\langle A', S\rangle$ is the set of all $(s+1)$-tuple $(r_0, \ldots, r_s)$ with $r_i \in S$. Then $S \preceq R_M$ implies (2). As mentioned above $\langle A, S\rangle$ is a $(\lambda_1, \ldots, \lambda_c)$-subalgebra of some $L_c(S, \kappa)$ with $\lambda_1 \geq \omega$. By Theorem 6.3 $\langle A, S\rangle$ models $\Delta_c(R)$. $\square$

Lemma 6.6. *Let M be a model of $\Delta_c(R)$ and H be a $\mathfrak{L}_2$-substructure of M. Let $A = \{a^i_{\alpha(i)}: 1 \leq i \leq c, \alpha(i) < \lambda_i\}$ be a generating (o)-system of H in H.*

If $2 \leq \lambda_1$, $R_H \preceq R_M$, and there is a generating (o)-system B of M extending A, Then $H \overset{*}{\preceq} M$.

On the other hand $H \overset{*}{\preceq} M$ implies the existence of a generating (o)-system B of M extending A.

Proof. First assume $2 \leq \lambda_1$, $R_H \preceq R_M$, and there is a generating (o)-system B of M extending A. By Lemma 6.5 $\langle B, R_H\rangle \overset{*}{\preceq} M$ and $\langle B, R_H\rangle$ models $\Delta_c(R)$. From Lemma 6.4 follows $H = \langle A, R_H\rangle \overset{*}{\preceq} \langle B, R_H\rangle \overset{*}{\preceq} M$, as desired.

Now suppose $H \overset{*}{\preceq} M$. We construct inductively an increasing chain of (o)-systems $B_j \subseteq M \setminus Z_{c-j}(M)$ such that B_j generates M modulo $Z_{c-j}(M)$ and $a^i_{\alpha(i)} \in B_j$ for $i \leq j$. Then $B_c = B$ is a generating (o)-system of M extending A.

Let $B_0 = \emptyset$. Assume B_{j-1} is constructed and has the desired properties. Then it suffices to show, that $B_{j-1} \cup \{a^j_{\alpha(j)} \in A: \alpha(j) < \lambda_j\}$ is an (o)-system in M, because it is easy to extend this set to some B_j with the desired properties.

If $B_{j-1} \cup \{a^j_{\alpha(j)}: \alpha(j) < \lambda_j\}$ would not be an (o)-system there would be $a_1, \ldots, a_m \in \{a^j_{\alpha(j)}: \alpha(j) < \lambda_j\}$ with $M \models U_{j,n}(a_1, \ldots, a_m)$ for some n. By $H \overset{*}{\preceq} M$ it follows $H \models U_{j,n}(a_1, \ldots, a_m)$ for some n. We show that this is impossible, since A is a generating o-system of H.

$H \models U_{j,n}(a_1, \ldots, a_m)$ means that there are an $(o)_{j,n}$-system $C = \{c^i_{\alpha(i)}: 1 \leq i \leq j, \alpha(i) < k_i\}$ $(k_i \leq n)$ and elements $r_1, \ldots, r_m$ of R_H, such that $r_i \neq 0$ for some i and

$$\sum_{1 \leq i \leq m} r_i a_i + \sum_l s_l d_l = 0 \quad \text{modulo } Z_{c-j}(H)$$

where $s_l \in R_H$ and the d_l's are basic monomials over C. For A is a generating (o)-system of H every c_k^i is a linear combination of basic monomials on

$$\{a^l_{\alpha(l)}: 1 \leq l \leq j, \alpha(l) < \lambda_l\} \quad \text{modulo } Z_{c-j}(H).$$

Therefore we can assume w.l.o.g. that $C \subseteq \{a^i_{\alpha(i)} \in A: i < j\}$, a contradiction. $\square$

Lemma 6.7. *If M is a λ-saturated model of $\Delta_c(R)$ and $A = \{a^i_{\alpha(i)}: 1 \leq i \leq c, \alpha(i) < \lambda_i\}$ is a generating (o)-system of M, then $\lambda_1 \geq \lambda$.*

Proof. Suppose there is some j such that $\lambda_i \geq \lambda$ for $i < j$, but $\lambda_j < \lambda$. Then $1 < j$. We shall show a contradiction. Let $p(x)$ be the following type

$$\{U_{j-1,0}(x)\} \cup \{\neg U_{j,n}(c_1, \dots, c_m, x): m, n < \omega, c_1, \dots, c_m \in \{a^j_{\alpha(j)}: \alpha(j) < \lambda_j\}\}$$

$p(x)$ is consistent by Lemma 2.15. Since $p(x)$ contains less than λ-many elements of M and M is λ-saturated, there is some $a \in M$, that realizes $p(x)$. By $(\Delta 3)$ A is a $(*)$-system in M. By Lemma 2.5 we get a contradiction, because a must be a linear combination of basic monomials on $\{a^i_{\alpha(i)}: 1 \leq i \leq j, \alpha(i) < \lambda_i\}$ of A-degree j modulo $Z_{c-j}(M)$. $\square$

Now we can prove:

Theorem 6.8. *$\Delta^*_c(R)$ admits the elimination of quantifiers.*

Proof. As well known the assertion is equivalent to the following (see e.g. [19]):
Every diagram of the following sort can be completed as shown:

$$(3) \qquad \begin{tikzcd} & K & \\ M \arrow[ur, dotted, "\equiv"] & & N \arrow[ul, dotted, "*"'] \\ & H \arrow[ul, "*"] \arrow[ur, "*"'] & \end{tikzcd} \qquad M, N \vDash \Delta^*_c(R).$$

Assume M, H, N are given as in (3) above. By construction $\mathrm{Th}^*(R)$ admits the elimination of quantifiers. Therefore there exists some S with

$$(4) \qquad \begin{tikzcd} & S & \\ R_M \arrow[ur, "\cong"] & & R_N \arrow[ul, "\cong"'] \\ & R_H \arrow[ul, "*"] \arrow[ur, "*"'] & \end{tikzcd}$$

$R_M \leq S$ implies the existence of some $M' \geq M$ such that $R_{M'} \geq S$. We define K to be a λ-saturated elementary extension of M', where λ is some cardinal greater than card(M') and card(N). From (4) follows

$$(5)\qquad \begin{tikzcd} & R_K & \\ R_M \arrow[ur, "\equiv"] & & R_N \arrow[ul, "\equiv"'] \\ & R_H \arrow[ul, "*"] \arrow[ur, "*"'] & \end{tikzcd}$$

Let A be a generating (o)-system of H in H. By Lemma 6.6 there are (o)-systems B and C generating M resp. N and containing A. By the same argument we get a generating (o)-system D of K that contains B.

Suppose

$$D = \{d^i_{\alpha(i)}: 1 \leqslant i \leqslant c, \alpha(i) < \lambda_i\},$$

$$B = \{d^i_{\alpha(i)} \in D: 1 \leqslant i \leqslant c, \alpha(i) < \kappa_i\}, \quad \text{where } \kappa_i \leqslant \lambda_i,$$

and

$$A = \{d^i_{\alpha(i)} \in B: 1 \leqslant i \leqslant c, \alpha(i) < \nu_i\}, \quad \text{where } \nu_i \leqslant \kappa_i.$$

Since K is a λ-saturated model of $\Delta_c(R)$ we obtain $\lambda_i \geqslant \lambda$ by Lemma 6.7. Furthermore we can assume

$$C = \{c^i_{\alpha(i)}: 1 \leqslant i \leqslant c, \alpha(i) < \mu_i\}$$

such that $\nu_i \leqslant \mu_i \leqslant \lambda_i$ and $c^i_{\alpha(i)} = d^i_{\alpha(i)} \in A$ for $\alpha(i) < \nu_i$. For $(\Delta 3)$ B, C, D, and therefore A are $(*)$-systems. We define $\chi(c^i_{\alpha(i)}) = d^i_{\alpha(i)}$. By Lemma 2.12 χ induces an isomorphism of N onto the substructure $\langle \chi(C), R_N \rangle$ of K. By construction $\chi(\langle A \rangle) = \langle A \rangle$. Since $R_K \geqslant R_N$ Lemma 6.6 implies $\langle \chi(C), R_N \rangle \overset{*}{\subseteq} K$. Using (5) the assertion (3) follows.

Corollary 6.9. *$\Delta_c(R)$ is complete.*

Corollary 6.10. *For every $\kappa \geqslant \omega$ and $R \equiv S$ $L_c(R, \omega) \equiv L_c(S, \kappa)$. Also the elementary theory of $L_c(R, \kappa)$ is given by $\Delta_c(R)$ and therefore recursive relative to* Th(R).

If T is a countable elementary theory, then the stability function $f_T(\kappa)$ of T is defined as follows. Let $\gamma(M)$ be the set of 1-types of the T-model M.

$$f_T(\kappa) = \sup(\{\text{card}(\gamma(M)): M \models T \text{ and card}(M) = \kappa\}).$$

Two theories (structures) are in the same stability class iff they (their theories) have the same stability function.

Theorem 6.11. *For every $\kappa \geqslant \omega$ and* card$(R) \geqslant \omega$ Th(R) *and $\Delta_c(R)$ have the same stability class. If* card$(R) < \omega$ *$\Delta_c(R)$ is ω-stable.*

This theorem follows from:

Theorem 6.12. *If M is a model of $\Delta_c(R)$, then* $\mathrm{card}(\gamma(M))=\max\{\mathrm{card}(M), \mathrm{card}(\gamma(R_M))\}$.

Proof. Let N be an elementary extension of M such that every 1-type over M is realized in N. We use the notation $\lambda=\max\{\mathrm{card}(M),\mathrm{card}(\gamma(R_M))\}$. Then there is some field S such that $R_M\leq S\leq R_N$, $\mathrm{card}(S)=\lambda$, and every 1-type over R_M is realized in S. By Lemma 6.6 there exists an (o)-system $A=\{a^i_{\alpha(i)}: 1\leq i\leq c, \alpha(i)<\lambda_i\}$ generating N and cardinals $\kappa_i\leq\lambda_i$, λ $(1\leq i\leq c)$ such that $A'=\{a^i_{\alpha(i)}\in A: \alpha(i)<\kappa_i\}$ is a generating (o)-system of M. By Lemma 6.6 and Theorem 6.8 $M=\langle A', R_M\rangle\leq\langle A', S\rangle\leq\langle A, S\rangle\leq\langle A, R_N\rangle=N$. Next we show

(6) If $p(x)$ is a 1-type over M realized in N, then $p(x)$ is realized in $\langle A, S\rangle$.

By Theorem 6.8 we can assume that $p(x)$ consists of unnegated and negated atomic formulas $\varphi(x)$ of $\mathfrak{L}^*_2$. Therefore (6) for field elements is clear. Otherwise let $\{b_\alpha: \alpha<\kappa\}$ be an enumeration of the basic monomials on A. Assume $N\models p(a)$ and $a=\sum_{0\leq i\leq m} r_i b_{\alpha_i}$ be the unique presentation of a. Let $q(x_0,\ldots,x_m)$ be the m-type of $(r_0,\ldots,r_m)$ over R_M. By construction of S there is some tuple $(s_0,\ldots,s_m)$ realizing $q(x_0,\ldots,x_m)$ in S. Our assertion is that $\sum_{0\leq i\leq m} s_i b_{\alpha_i}$ realizes $p(x)$ in $\langle A, S\rangle$. Let $\varphi(x)$ be a formula of $p(x)$. Let C be a finite subset of A such that a and all parameters of $\varphi(x)$ are in $\langle C\rangle$, and a^1_0 and a^1_1 are in C. By Lemma 6.6 $\langle C, R_N\rangle\stackrel{*}{\leq} N$. Therefore $\langle C, R_N\rangle\models\varphi(a)$. We use $b'_0,\ldots,b'_n$ to denote the basic monomials on C. Then every element c of $\langle C, R_N\rangle$ has a unique presentation $\sum_{0\leq i\leq n} t_i b'_i$ with $t_i\in R_N$. $c\in\langle C, S\rangle$ iff $t_i\in S$. We can interpret $\langle C, R_N\rangle$ in R_N coding the elements as $(n+1)$-tuple as in the proof of Lemma 6.5. There is some φ^* of $\mathfrak{L}^*_f$ such that for every $S<S'\leq R_N$ and every $u_0,\ldots,u_n\in S'$

$$\langle C, S'\rangle\models\varphi\Big(\sum_{0\leq i\leq n} u_i b'_i\Big)\quad\text{iff } S'\models\varphi^*(u_0,\ldots,u_n).$$

W.l.o.g. we can assume $n=m$, $b'_i=b_{\alpha_i}$ and therefore $a=\sum_{1\leq i\leq n} r_i b'_i$. Then $R_N\models\varphi^*(r_0,\ldots,r_n)$ and $S\leq R_N$ implies $S\models\varphi^*(s_0,\ldots,s_n)$ by the choice of $s_0,\ldots,s_n$. Hence $\langle C, S\rangle\models\varphi(\sum_{0\leq i\leq n} s_i b_{\alpha_i})$ as desired in (6).

Now we finish the proof of the theorem showing that for every $a\in\langle A, S\rangle$ there is an automorphism χ of $\langle A, S\rangle$ such that χ is the identity on $\langle A'\rangle$ and

$$\chi(a)\in\langle\{a^i_{\alpha(i)}: 1\leq i\leq c, \alpha(i)<\kappa_i+\omega\}, S\rangle.$$

Suppose there are some n and some $\gamma_{i,j}\geq\kappa_i$ $(1\leq i\leq c, j<n)$ such that $a\in\langle A'\cup\{a^i_{\gamma_{i,j}}: 1\leq i\leq c, j<n\}, S\rangle$. By Theorem 6.3 and Lemma 6.5 $\langle A, S\rangle$ is a $(\lambda_1,\ldots,\lambda_c)$-substructure of some $L_c(S,\kappa)$. By Corollary 2.6(ii) every permutation $\mathfrak{S}$ of A with $\mathfrak{S}(a^i_{\alpha(i)})=a^i_\gamma$ for some γ is again an (o)-system in $L_c(S,\kappa)$. Hence by

Lemma 2.9

$$\chi(a^i_{\alpha(i)}) = \begin{cases} a^i_{\alpha(i)} & \text{if } \alpha(i) \neq \kappa_i + j \text{ and } \alpha(i) \neq \gamma_{i,j} \text{ for all } j < n, \\ a^i_{\kappa_i + j} & \text{if } \alpha(i) = \gamma_{i,j} \text{ for some } j < n, \\ a^i_{\gamma_{i,j}} & \text{if } \alpha(i) = \kappa_i + j \text{ for some } j < n, \end{cases}$$

induces the desired automorphism of $\langle A, S\rangle$.

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